LETTER TO THE EDITOR

Non-equilibrium critical behaviour on fractal lattices

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Abstract. A simple one-component surface reaction model exhibiting a critical point has been studied on deterministic fractal lattices with fractal dimension $1 < D_f < 2$. Results from time-dependent simulations show a change in the value of the critical exponents with D_f. Two of the exponents interpolate very nicely between the values in one and two dimensions.

Non-equilibrium phase transitions have been studied intensively in recent years. Special attention has been devoted to the question of universality classes. A large variety of non-equilibrium models such as the contact process [1, 2], Schlögl's first model [3–5], directed percolation [6–8], Reggeon field theory [9, 4] and the zGB model [10, 11], all of which exhibit a continuous transition from an active state to an absorbing state, have the same critical behaviour. Studies of related models via computer simulations [12–15], field theoretic arguments [16, 17] and series expansions [18] demonstrate the robustness of this universality class against a wide range of changes in the local kinetic rules such as multi-particle processes, diffusion and changes in the number of chemical components. So currently there is substantial evidence in favour of the conjecture [5, 16, 17] that models with a scalar order parameter exhibiting a continuous transition to an absorbing state generically belong to the universality class of directed percolation/Reggeon field theory. So far it seems that the critical exponents depend solely on the dimensionality. Studies of fractal spin systems, e.g. Ising-spins placed on fractal lattices [19], have shown that the critical exponents depend not only on the fractal dimensionality but also on several topological factors: connectivity, ramification, lacunarity etc [19]. Motivated by this I have studied this type of non-equilibrium critical behaviour on fractal lattices.

In this letter I study a simple one-component non-equilibrium lattice model with spontaneous annihilation and autocatalytic creation of particles. Each site can be either vacant or occupied by a single particle. The state of the lattice is thus given by a set of occupation variables $\{\sigma_i\}$, where $\sigma_i = 0$ (1) when site i is vacant (occupied). The system evolves in time through a sequence of changes each involving a single elementary process. The annihilation process, which takes place with probability p, simply turns an occupied site into a vacant site. With probability $1 - p$ an occupied site creates a new particle autocatalytically at a randomly chosen vacant nearest neighbour. As there is no spontaneous creation of particles the state with all sites vacant (the vacuum state) is an absorbing state, i.e. a state that the system cannot possibly leave.

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The processes always involve an occupied site, so by maintaining a list of these sites the updating in a sequential Monte Carlo simulation can be made very efficient. After each attempted process time is incremented by $1/N_{occ}$ where N_{occ} is the number of occupied sites. In the long time limit the system reaches a steady state characterized by the average concentration of occupied sites ρ, which is the appropriate order parameter. When the annihilation probability p exceeds a critical value p_c all particles will eventually disappear from the lattice and the system enters the absorbing state. Below p_c an active state with a non-zero average concentration of particles is reached (in the infinite size limit). The order parameter ρ changes continuously at p_c. The non-equilibrium phase transition at p_c is thus a continuous transition from an active state to an absorbing state.

I have studied the model on two determinstic fractal lattices namely the Sierpinski carpet and the Tamás Vicsek fractal. The Sierpinski carpet is generated by removing, in each step, the middle 1/9 of all squares and it thus has the fractal dimension [20] $D_f = \ln 8/\ln 3 = 1.8927 \ldots$. The Tamás Vicsek fractal is generated by removing, in each step, the four squares in the middle of each side leading to a fractal with dimension [20] $D_f = \ln 5/\ln 3 = 1.4649 \ldots$. The two fractals are shown schematically in figure 1. The Sierpinski carpet is connected along the usual nearest neighbours whereas the Tamás Vicsek fractal is connected along the diagonals or next-nearest neighbours of an ordinary two-dimensional square lattice. In the simulations I used a Sierpinski carpet with six levels giving a linear size $L = 3^6 = 729$ and a number of sites belonging to the fractal $N = 8^6 = 262\,144$. Before making any simulations the Sierpinski carpet, shown in figure 1, was translated so that the uppermost left corner was placed at the centre (the reason for this should become clear, I hope, in a short while). The Tamás Vicsek fractal had seven levels and thus a linear size $L = 3^7 = 2187$ and the number of sites in the fractal was $N = 5^7 = 78\,125$.
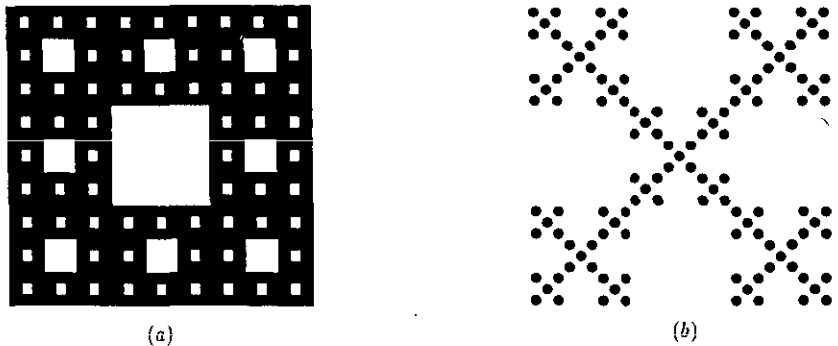

(a)

(b)

Figure 1. The Sierpinski carpet (a) and the Tamás Vicsek fractal (b). Both fractals are shown with three levels (sizes of holes). On the Sierpinski carpet each black square with the size of the smallest holes represent a single site. Each site on the Tamás Vicsek fractal is represented by a black dot.

In order to determine some of the critical exponents I used time-dependent simulations, which have proved to be a very efficient method for determining critical points and exponents [4, 8, 11, 13]. The general idea is to start from the configuration closest to the absorbing state, in this case a configuration with a single seed particle, and then follow the 'average' time evolution of the configuration. From this initial configuration

I made a number of independent runs, typically $1-2 \times 10^5$, for different values of p in the vicinity of p_c. Each run had a maximal duration of 5000 timesteps. As fractals are not homogeneous objects the results might depend on the environment of the seed particle. Starting out on the edge of a large hole is surely different from starting in an area with a high density of sites. In order to reduce this effect I made an average over different initial positions of the seed particle. I did this by choosing in each run a random position for the seed particle in the middle 1/9 of the fractal. I measure the survival probability $P(t)$ (the probability that the system had not entered the absorbing state at time t), the average number of occupied sites $n(t)$ and the average mean square distance of spreading $R^2(t)$ from the position of the seed. It should be noticed that $n(t)$ is averaged over all runs whereas $R^2(t)$ is averaged only over the surviving runs. From the scaling ansatz for the contact process and similar models [4, 8] it follows that the quantities defined above are, in ordinary (non-fractal) D-dimensional spaces, governed by power laws at p_c at $t \to \infty$

$$P(t) \propto t^{-\delta} \tag{1}$$

$$n(t) \propto t^{\eta} \tag{2}$$

$$R^2(t) \propto t^z. \tag{3}$$

The three exponents δ, η, and z are related through the scaling relation

$$Dz = 4\delta + 2\eta. \tag{4}$$

As will become clear, the same relations are true for fractal lattices.

In log-log plots of $P(t)$, $n(t)$ and $R^2(t)$ versus t we should see asymptotically a straight line at $p = p_c$. The curves will show positive (negative) curvature when $p < p_c$ ($p > p_c$). This makes it possible to obtain precise estimates for p_c. The asymptotic slope of the (critical) curves define the dynamic critical exponents δ, η and z. Generally we have to expect corrections to the pure power law behaviour so that $P(t)$ is more accurately given by the formula [8]

$$P(t) \propto t^{-\delta}(1 + at^{-1} + bt^{-\delta'} + \ldots) \tag{5}$$

and similarly for $n(t)$ and $R^2(t)$. More precise estimates for the critical exponents can be obtained if one looks at local slopes, defined as

$$-\delta(t) = \frac{\ln[P(t)/P(t/m)]}{\ln(m)} \tag{6}$$

and similarly for $\eta(t)$ and $z(t)$. The local slope $\delta(t)$ behaves as [8]

$$\delta(t) = \delta + at^{-1} + b\delta' t^{-\delta'} + \ldots \tag{7}$$

and similar expressions for $\eta(t)$ and $z(t)$. Thus in a plot of the local slopes versus $1/t$ the critical exponents are given by the intercept with the y-axis.

The results for $P(t)$, $n(t)$, and $R^2(t)$ on the Sierpinski carpet are shown in figure 2(a) and the corresponding local slopes $-\delta(t)$, $\eta(t)$ and $z(t)$ are shown in figure 2(b). The curves for $n(t)$ and $\eta(t)$ clearly deomonstrate that both $p = 0.4572$ and $p = 0.4570$ are off-critical leading to a very precise estimate of the critical point $p_c = 0.457\,10 \pm 0.000\,05$. the similar results for the Tamás Vicsek fractal are shown in figures 3(a) and 3(b). In this case the distinction between the different values of p is not so clear-cut. The results for $\eta(t)$ does, however, strongly indicate that the critical point is $p_c = 0.3463 \pm 0.0001$. The estimtes for the critical exponents δ, η and z are stated in table 1, which also shows the values obtained on ordinary one- and two-dimensional lattices [4, 8].
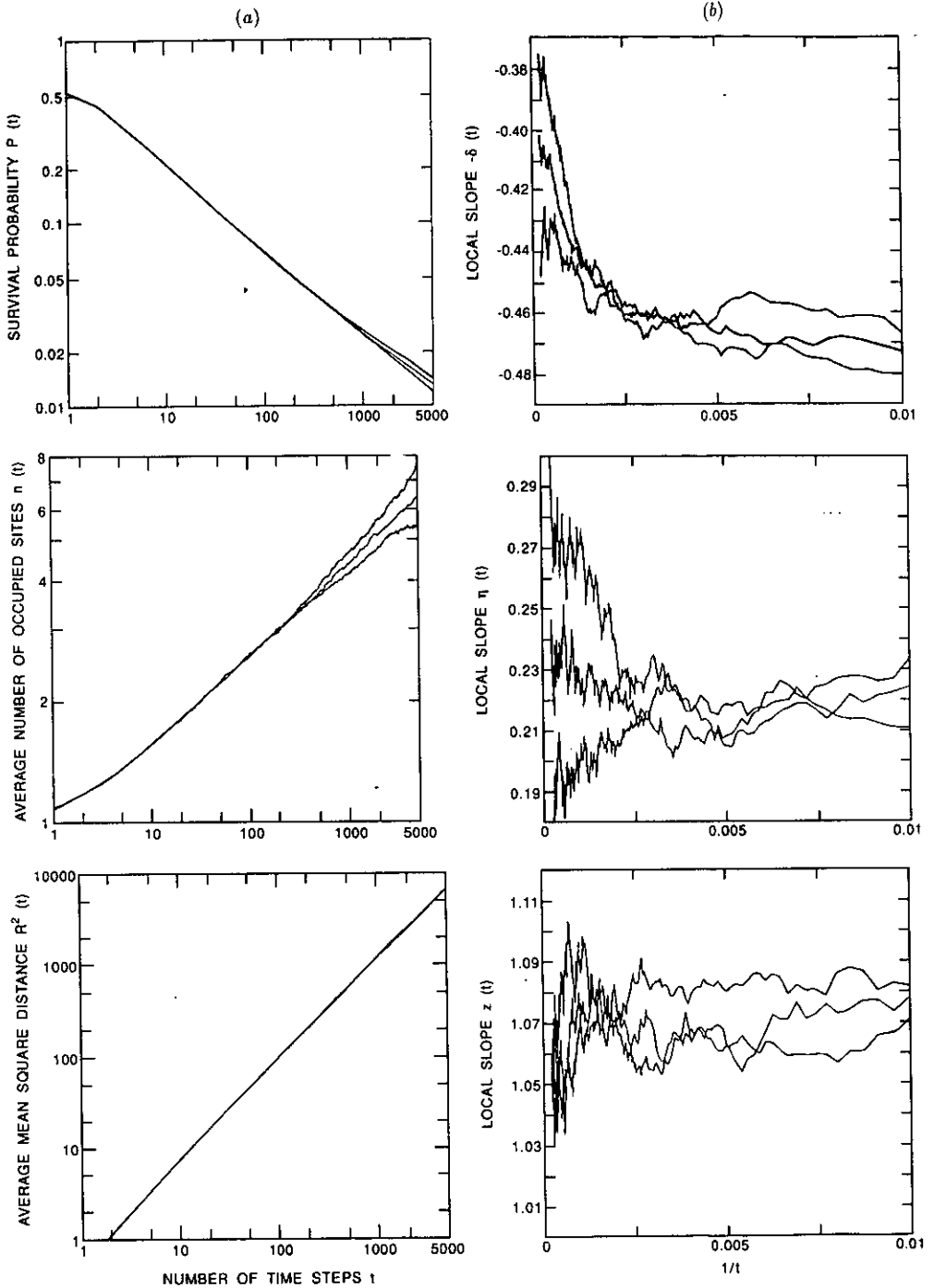
Figure 2. (*a*) Log–log plot of the survival probability (upper panel), the average number of occupied sites (middle panel), and the average mean square distance of spreading (lower panel) for the Siepinski carpet. Each panel contains three curves with, from bottom to top, $p = 0.4572, 0.4571$, and 0.4570. (*b*). Local slopes $-\delta(t)$ (upper panel), $\eta(t)$ (middle panel), and $z(t)$ (lower panel), as defined in (6) with $m = 5$, for the curves in figure 2(*a*).

Figure 3. (*a*) Log–log plot of the survival probability (upper panel), the average number of occupied sites (middle panel), and the average mean square distance of spreading (lower panel) for the Tamás Vicsek fractal. Each panel contain three curves with, from bottom to top, $p = 0.3464$, 0.3463, and 0.3462. (*b*) Local slopes $-\delta(t)$ (upper panel), $\eta(t)$ (middle panel), $z(t)$ (lower panel), As defined in (6) with $m = 5$, for the curves in figure 3(*a*).
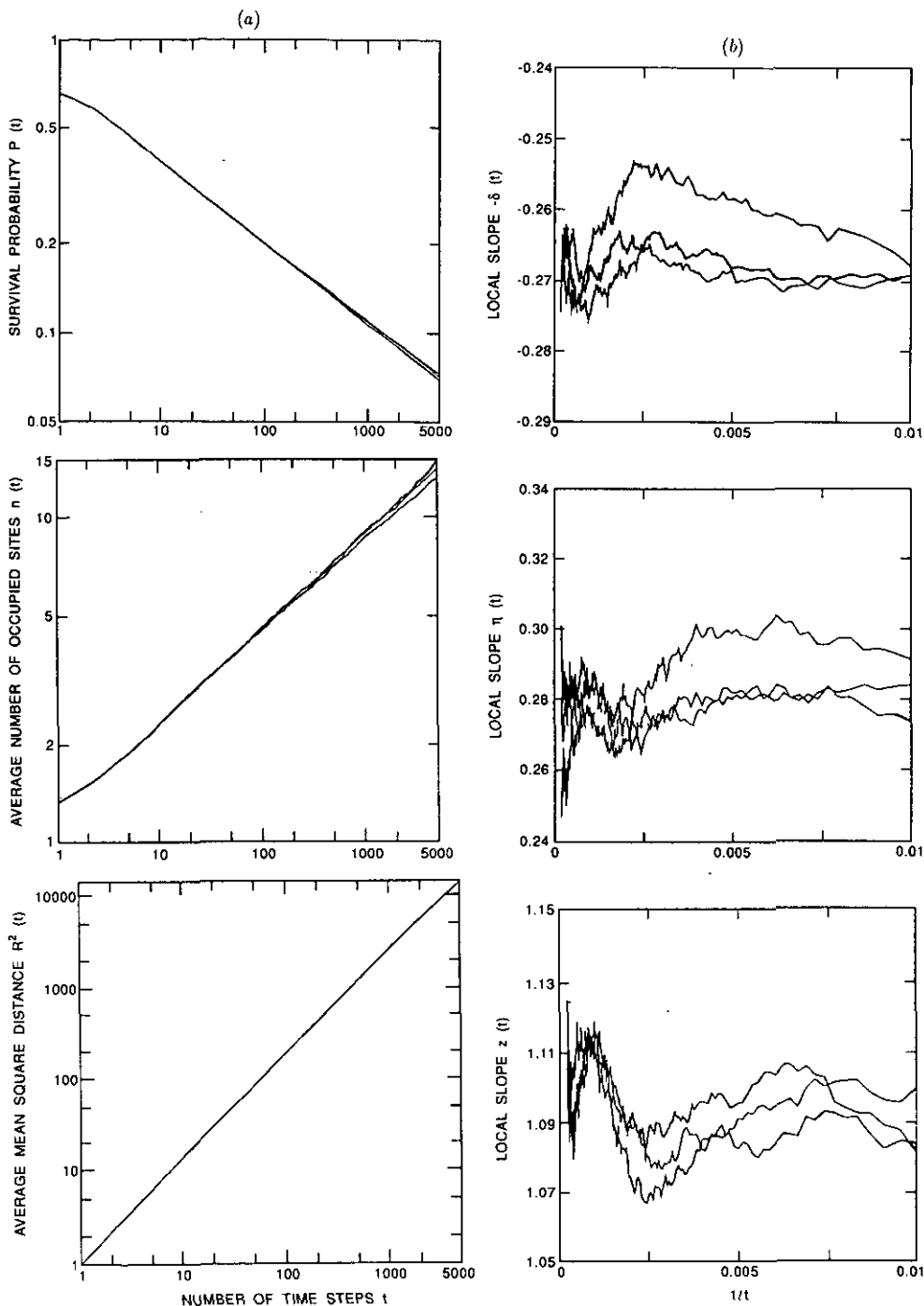
Table 1. Critical exponents δ, η, and z on, from top to bottom, one-dimensional lattices [4], the Tamás Vicsek fractal, the Sierpinski carpet, and two-dimensional lattices [8].

D_f	δ	η	z
1	0.162 ± 0.004	0.308 ± 0.009	1.263 ± 0.008
$\ln 5/\ln 3$	0.265 ± 0.005	0.285 ± 0.010	1.100 ± 0.010
$\ln 8/\ln 3$	0.400 ± 0.010	0.235 ± 0.010	1.060 ± 0.015
2	0.460 ± 0.006	0.214 ± 0.008	1.134 ± 0.004

The values of δ and η obtained on the Tamás Vicsek fractal and the Sierpinski carpet interpolate very nicely between the values on one- and two-dimensional lattices. There is, however, a significant deviation from this picture when we look at the critical exponent z for the average mean square distance of spreading. This deviation might be due to the way I measured the distance from the initial seed, as I simply used the Euclidean distance. It is questionable whether this is a correct procedure for measuring the distance between points on a fractal. Two points on a fractal which are close in Euclidean space might very well be far apart in the sense that the minimal length taken over all the paths connecting the points can be much larger than their Euclidean distance. In fact one probably ought to use this measure of distance instead of the Euclidean distance.

The results for z could, however, be a major sign that new effects come into play when considering fractal lattices. From renormalization group calculations the ε-expansion [21, 22] yields exponents which depends monotonically on $\varepsilon = 4 - D$, when no account is taken of the special geometry of a fractal†. As shown by Gefen *et al* [19], a variety of geometrical factors have a profound influence on the critical exponents of fractal spin systems. A similar influence should probably be expected for the kind of non-equilibrium phase transitions studied in this letter. Significant evidence in favour of this interpretation of the simulation results can be found in the fact that the scaling relation (4) still seems to hold true. Inserting the values for the critical exponents and D_f one finds for the Tamás Vicsek fractal: $4\delta + 2\eta = 1.630 \pm 0.040$ and $D_f z = 1.611 \pm 0.0015$, and for the Sierpinski carpet: $4\delta + 2\eta = 2.070 \pm 0.060$ and $D_f z = 2.006 \pm 0.028$. Although the agreement is somewhat marginal for the Sierpinski carpet there is strong evidence that the scaling relation (4) still hold true in these cases. This in turn indicates that the special behaviour of z is generic to the two systems studied and not an artefact of the way distances are measured.

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† To second order in ε the exponents are: $\eta = \frac{1}{12}\varepsilon + (\frac{161}{12}\ln\frac{4}{3} + \frac{37}{24})(\frac{1}{12}\varepsilon)^2 + \ldots$, and $z = 1 + \frac{1}{24}\varepsilon + (\frac{59}{24}\ln\frac{4}{3} + \frac{79}{48})(\frac{1}{12}\varepsilon)^2 + \ldots$.

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